

NONCOMMUTATIVE GEOMETRY AND QUANTIZATION

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We examine some recent developments in noncommutative geometry, including spin geometries on noncommutative tori and their quantization by the Shale–Stinespring procedure, as well as the emergence of Hopf algebras as a tool linking index theory and renormalization calculations.

Introduction

The purpose of these lectures is to survey several aspects of noncommutative geometry, with emphasis on its applicability to particle physics and quantum field theory. By now, it is a commonplace statement that spacetime at short length—or high energy—scales is not the Cartesian continuum of macroscopic experience, and that older methods of working with functions on manifolds must be rethought in order to handle granular or bubbly spacetime and the matter fields which they support. The question as to what should replace the continuum is an ongoing one. Noncommutative geometry (NCG) offers a general approach to the job of describing the geometrical aspects of nonclassical spaces. The hope is that it can provide a solid framework for studying fundamental interactions and QFT.

Here we consider four aspects of this general problem. In Section 1, we discuss the noncommutative tori that have recently emerged as an important model in string theory.^{1–3} Section 2 reviews the general structure of noncommutative geometries based on the spectral triples of Connes.⁴ The third section broaches the formulation of quantum field theories over noncommutative spaces, using the example of a 3-torus.^{5,6} In Section 4, we sketch how Hopf algebras provide a link between NCG and renormalization calculations,^{7–10} that illuminates the claim that “QFT is the geometry of the world”.

1. Noncommutative tori

The spaces and tools of NCG can be approached from two different points of view: either as theoretical constructs that form part of a principled explanatory scheme, or as objects that are fortuitously “found in Nature” and therefore point to an (as yet incomplete) underlying theory. A striking example of the second viewpoint is the emergence of noncommutative tori from compactification of Matrix models.¹

In those models, one considers an action functional of the general form

$$I = \sum_{\alpha,\beta} \text{Tr} F_{\alpha\beta}^2 + 2 \sum_{i,j,\alpha} \text{Tr} \Psi^i \gamma_{ij}^\alpha [\nabla_\alpha, \Psi^j],$$

where ∇ is a connection and F is its curvature. For instance, we may ask that $\nabla_\alpha = X_\alpha$ be matrix components of some Lie algebra representation, $F_{\alpha\beta} = [X_\alpha, X_\beta]$, the γ^α being gamma-matrices, and the Ψ^i are odd variables. In the example considered in Ref. 1, the matrices (X_α, Ψ^i) are labelled by coordinates of the superspace $\mathbb{C}^{10|16}$.

To compactify, we ask that at least some of the X_α variables change only by a gauge transformation under certain fixed translations $X_\alpha \mapsto X_\alpha + r_\alpha$. For two such directions, this gives

$$\begin{aligned} X_0 + r_0 &= u_0 X_0 u_0^{-1}, \\ X_1 + r_1 &= u_1 X_1 u_1^{-1}, \\ X_\alpha &= u_\beta X_\alpha u_\beta^{-1} && \text{in all other cases,} \\ \Psi^i &= u_\alpha \Psi^i u_\alpha^{-1} && \text{in all cases,} \end{aligned}$$

for certain unitary operators u_α (to be determined). By taking traces of the first two equations, it becomes clear that these equations have no solutions in $N \times N$ matrices. However, there are formal operator solutions, such as

$$X_0 = ir_0 \frac{\partial}{\partial \phi_0} + A_0(\phi_0, \phi_1), \quad X_1 = ir_1 \frac{\partial}{\partial \phi_1} + A_1(\phi_0, \phi_1),$$

and $X_\alpha = A_\alpha(\phi_0, \phi_1)$ for other α , where ϕ_0, ϕ_1 are angular variables and u_0, u_1 are rotations of these angles.

To get a clearer picture, notice that *all* X_α and Ψ^i commute with the unitary $u_1 u_0 u_1^{-1} u_0^{-1}$. If we are looking for an irreducible solution of the equations, we can suppose that this unitary is a scalar. Thus there is some number $\lambda = e^{2\pi i \theta}$ of absolute value 1, such that

$$u_1 u_0 = e^{2\pi i \theta} u_0 u_1. \quad (1)$$

Now, the point is that nothing we have said so far demands that λ be equal to 1 (that is, $\theta = 0$), so we can perfectly well suppose that it is not. Indeed, if $\theta = 0$, then u_0 and u_1 can be taken as coordinate rotations on the ordinary torus \mathbb{T}^2 . For any other (non-integer) value of θ , u_0 and u_1 become “coordinates” for a certain noncommutative space, namely, the NC torus \mathbb{T}_θ^2 .

When compactifying in more than two dimensions, we obtain more relations, of the form

$$u_k u_j = e^{2\pi i \theta_{jk}} u_j u_k, \quad (2)$$

where $\theta = [\theta_{jk}]$ is now a real skewsymmetric matrix.

1.1. The NC torus as a noncommutative space

The last statement is to be interpreted as follows. Noncommutative topology replaces a locally compact space of points Y by its algebra $C_0(Y)$ of continuous functions vanishing at infinity; this is a commutative C^* -algebra. No information is lost (or gained) in passing from the space to the algebra, and the process is reversible: this is the content of the Gelfand–Naimark theorem.¹¹ For instance, the space is compact if and only if the algebra has a unit element, the space is disconnected if and only if the algebra contains nontrivial idempotents, and so on. To deal with gauge potentials and suchlike, it is often better to work instead with the dense subalgebra of smooth functions. Finally, we abandon points by discarding the commutativity property and by calling any C^* -algebra a “noncommutative space”; or, if smoothness remains important, we work with certain dense subalgebras called pre- C^* -algebras.

From this point of view, the NC torus \mathbb{T}_θ^2 is just a certain dense subalgebra of the C^* -algebra generated by two unitaries subject only to the relation (1). Notice that this is not the notorious “quantum plane”, since the requirement of unitarity is met by four more relations: $u_0 u_0^\dagger = u_0^\dagger u_0 = 1$, $u_1 u_1^\dagger = u_1^\dagger u_1 = 1$, which are not asked of the quantum plane.

The n -dimensional noncommutative tori are to be regarded as quantizations of the ordinary torus, whose algebra is $C^\infty(\mathbb{T}^n) =: \mathbb{T}_0^n$. Indeed, noncommutativity is achieved by replacing the ordinary product of functions on \mathbb{T}^n with the Moyal product: see, for instance, Ref. 12 or our Ref. 13. A quick-and-dirty way to do this is by describing elements of \mathbb{T}_θ^n as “noncommutative Fourier series”.

For definiteness, we take $n = 3$ and let θ be a real skewsymmetric 3×3 matrix. The 3-torus is generated by three unitaries u_1, u_2, u_3 , subject to the commutation relations (2) and no others. Introduce the unitary Weyl elements^{5,14}

$$u^r := \exp\{\pi i(r_1 \theta_{12} r_2 + r_1 \theta_{13} r_3 + r_2 \theta_{23} r_3)\} u_1^{r_1} u_2^{r_2} u_3^{r_3},$$

for each $r \in \mathbb{Z}^3$; the coefficient is chosen so that $(u^r)^* = u^{-r}$ in all cases. They obey the product rule

$$u^r u^s = \lambda(r, s) u^{r+s}, \quad \lambda(r, s) := \exp\{-\pi i \sum_{j,k} r_j \theta_{jk} s_k\}. \quad (3)$$

Notice that $|\lambda(r, s)| = 1$ and $\lambda(r, \pm r) = 1$ by skewsymmetry of θ . This σ is in fact a 2-cocycle for the abelian group \mathbb{Z}^3 and the C^* -algebra $C^*(\mathbb{Z}^3, \sigma)$ is generated by the u^r subject to this product rule is called a twisted group C^* -algebra. The NC torus \mathbb{T}_θ^3 is the dense subalgebra consisting of all Fourier series

$$\mathbb{T}_\theta^3 := \{a = \sum_r a_r u^r : a_r \rightarrow 0 \text{ rapidly}\},$$

where rapid decrease of the coefficients means that $(1 + |r|^2)^k |a_r|^2$ is bounded for all $k = 1, 2, 3, \dots$. In the commutative case $\theta = 0$, this condition gives $\mathbb{T}_0^3 \simeq C^\infty(\mathbb{T}^3)$.

This algebra is naturally represented on a certain Hilbert space, using an old trick (the GNS construction) that requires a faithful state on the algebra. In fact, there is one such state which is also a *trace*:

$$\tau(\sum_r a_r u^r) := a_0.$$

By completing \mathbb{T}_θ^3 in the Hilbert norm

$$\|a\|_2 := \sqrt{\tau(a^*a)} = (\sum_r |a_r|^2)^{1/2},$$

we get a Hilbert space \mathcal{H}_τ . We write \underline{c} for an element $c \in \mathbb{T}_\theta^3$ regarded as a vector in \mathcal{H}_τ . Then the GNS representation of \mathbb{T}_θ^3 is just

$$\pi(a) : \underline{c} \mapsto \underline{ac}.$$

In other words, \mathbb{T}_θ^3 acts on \mathcal{H}_τ by left multiplication operators.

The Lie group \mathbb{T}^3 acts by rotations on the algebra \mathbb{T}_θ^3 , as follows: for each r , $(z_1, z_2, z_3) \cdot u^r := z_1^{r_1} z_2^{r_2} z_3^{r_3} u^r$. The trace τ is invariant under this action; indeed, τ picks out the only rotation-invariant component of an element of \mathbb{T}_θ^3 .

There is a special *antiunitary* operator J_0 on \mathcal{H}_τ , given by

$$J_0(\underline{a}) := \underline{a}^*.$$

This is the Tomita conjugation¹⁵ determined by the cyclic and separating vector $\underline{1}$ for the representation π ; clearly, $J_0^2 = 1$. The operator

$$\pi^\circ(b) := J_0 \pi(b^*) J_0 : \underline{c} \mapsto J_0 \underline{b^* c^*} = \underline{cb}$$

is a *right* multiplication by the element b , and is an antirepresentation of \mathbb{T}_θ^3 . Equivalently, π° is a true representation of the “opposite algebra” of \mathbb{T}_θ^3 , obtained by reversing the product. A glance at (2) shows that this opposite algebra is $\mathbb{T}_{-\theta}^3$. Life is simpler if we forget to write the π ’s; the commutativity of left and right multiplications is then expressed as

$$[a, J_0 b^* J_0] = 0 \quad \text{for all } a, b. \quad (4)$$

Differential calculus on tori begins with the partial derivatives

$$\delta_j(\sum_r a_r u^r) := 2\pi i \sum_r r_j a_r u^r \quad (j = 1, 2, 3).$$

To see why they are partial derivatives, pretend that $\theta = 0$, so that the $u^r = \exp\{2\pi i(r_1\phi_1 + r_2\phi_2 + r_3\phi_3)\}$ are a basis for an (ordinary) Fourier expansion on \mathbb{T}^3 . As operators on the algebra, the δ_j are symmetric derivations:

$$\delta_j(ab) = (\delta_j a)b + a(\delta_j b), \quad \delta_j(a^*) = (\delta_j a)^*,$$

and they satisfy $\tau(\delta_j a) = 0$.

With the canonical trace τ and the partial derivatives, we build up certain rotation-invariant multilinear forms on \mathbb{T}_θ^3 . Besides τ itself, we find

$$\begin{aligned}\psi_j(a, b) &:= \tau(a \delta_j b), \\ \varphi_{jk}(a, b, c) &:= \frac{1}{2\pi i} \tau(a \delta_j b \delta_k c - a \delta_k b \delta_j c) \\ \omega(a, b, c, d) &:= -\frac{1}{4\pi^2} \varepsilon^{ijk} \tau(a \delta_i b \delta_j c \delta_k d).\end{aligned}$$

When $\theta = 0$, these forms correspond to the homological structure of the ordinary 3-torus: think of the vertex, the edges, the faces and the interior of a periodic box in 3-space. For instance, $\psi_j(a, b)$ is matched with the line integral of the 1-form $a db$ over the j th edge of the box. A deep theorem of Connes¹⁶ relates the de Rham homology of a (commutative) manifold M to the cyclic cohomology of the algebra $C^\infty(M)$: after factoring out certain redundancies, the resulting “periodic cyclic cohomology” of $C^\infty(M)$ is isomorphic to the de Rham homology of M . It turns out that τ , ψ_j , φ_{jk} and ω are cyclic cocycles, of respective degrees 0, 1, 2 and 3, for the torus \mathbb{T}_θ^3 , independently of θ .

There is also an algebraic counterpart of de Rham cohomology. There is a theory of cyclic homology of algebras,¹⁷ but we only need the (less complicated) Hochschild homology, and only in the top degree. The chains for this homology theory are multiple tensor products of algebra elements. Here, for example, is the algebraic counterpart of the *volume form* on a 3-torus:

$$\mathbf{c} := \frac{1}{6(2\pi i)^3} \varepsilon^{ijk} u_i^{-1} u_j^{-1} u_k^{-1} \otimes u_k \otimes u_j \otimes u_i.$$

The boundary operator b collapses one tensor product to a multiplication, yielding the following alternating sum:

$$b(t \otimes u \otimes v \otimes w) = tu \otimes v \otimes w - t \otimes uv \otimes w + t \otimes u \otimes vw - wt \otimes u \otimes v,$$

and an easy calculation⁵ shows that $b\mathbf{c} = 0$, so that \mathbf{c} is a Hochschild 3-cycle.

1.2. Spin geometry on the NC torus

Geometry now enters the picture through the *Dirac operator*

$$D = -i \gamma^\mu \delta_\mu$$

where the (Euclidean) gamma matrices satisfy $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \delta^{\mu\nu}$. When $n = 2m$ or $2m + 1$, these act on a vector space of dimension 2^m . The *spinor space* is then

$$\mathcal{H} := \mathcal{H}_\tau \oplus \mathcal{H}_\tau \oplus \cdots \oplus \mathcal{H}_\tau \quad (2^m \text{ times}).$$

In the cases $n = 2$ or 3 , there are two copies of \mathcal{H}_τ and the gamma matrices are just the standard Pauli matrices. For $n = 3$,

$$D := -i(\sigma_1 \delta_1 + \sigma_2 \delta_2 + \sigma_3 \delta_3) = -i \begin{pmatrix} \delta_3 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & -\delta_3 \end{pmatrix}. \quad (5)$$

For the case $n = 2$, there is no δ_3 and the diagonal entries are replaced by zeros, so D is an odd matrix. Indeed, the grading operator $\gamma^3 := -i\gamma^1\gamma^2$ anticommutes with D when $n = 2$. For $n = 3$, there is no grading available. When $\theta = 0$, we recover the well known Dirac operators on tori (with the standard flat metric and untwisted spin structure).

The Riemannian distance on the ordinary torus \mathbb{T}^n is determined by the Dirac operator, through the formula¹⁸

$$d(p, q) = \sup\{ |a(p) - a(q)| : a \in C(\mathbb{T}^n), \|[D, a]\| \leq 1 \}.$$

See Ref. 13 or Section VI.1 of Ref. 19 for a proof of that; the formula works because $\|[D, a]\|$ is the sup norm of the gradient of a . The formula also makes sense, as a definition, for noncommutative tori, with $C(\mathbb{T}^n)$ replaced by the C^* -algebra completion of \mathbb{T}_θ^n and p, q interpreted as pure states of this C^* -algebra. Unfortunately, that is of little use since the state space is quite complicated. The issues of using the distance formula for noncommutative algebras are thoroughly discussed by Rieffel in Ref. 20.

The *charge conjugation* operator on \mathcal{H} is given by

$$C := -i\gamma^2 \otimes J_0 = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix},$$

for $n = 2$ or 3 . For higher n , the first factor in C is a suitable product of gamma matrices that makes C antiunitary and satisfies $C\gamma^\mu C^{-1} = -\gamma^\mu$. In general, $C^2 = \pm 1$, with a dimension-dependent sign. In the commutative case, the factor J_0 reduces to complex conjugation of functions.

By combining several of these ingredients, we can represent Hochschild chains on the spinor spaces. An r -chain with values in $\mathbb{T}_\theta^n \otimes \mathbb{T}_{-\theta}^n$ is a sum of terms of the form $(a \otimes b^\circ) \otimes a_1 \otimes \cdots \otimes a_r$ where $b^\circ = \pi^\circ(b)$ lies in the opposite algebra; such a term is represented by the operator

$$\pi_D((a \otimes b^\circ) \otimes a_1 \otimes \cdots \otimes a_n) := aCb^*C^{-1}[D, a_1] \cdots [D, a_n].$$

When $b = 1$, we omit it. For example, if $n = 3$, we see that $[D, u_j] = 2\pi\sigma_j$, so the volume form is represented by

$$\pi_D(\mathbf{c}) = \frac{(2\pi)^3}{6(2\pi i)^3} \varepsilon^{ijk} \sigma_k \sigma_j \sigma_i = \frac{(2\pi)^3}{6(2\pi i)^3} (-6i) = 1.$$

When $n = 4$, the analogous calculation gives $\pi_D(\mathbf{c}) = \gamma_5$, from the product of four gamma matrices. In general, the volume form is represented by 1 or by the grading operator, according as n is odd or even.

The dimension may also be extracted from the Dirac operator, by examining the growth of its spectrum. Ignoring zero modes (of which there are only a finite number), we find that D^{-1} is a compact selfadjoint operator, so that both D and

$|D| := \sqrt{D^\dagger D}$ have discrete spectra. If $s_k(D)$ denotes the k th singular value of D (i.e., the k th eigenvalue of $|D|$) in increasing order, there is one and only one integer n for which

$$\sum_{k \leq N} s_k(D)^{-n} \sim C \log N \quad \text{as } N \rightarrow \infty.$$

This n is the *classical dimension* of the NC torus. Write $C =: \mathfrak{f}|D|^{-n}$ for the coefficient of logarithmic divergence. What happens is that, just as in calculations of the dimension of fractals, there is one critical value of n with $0 < \mathfrak{f}|D|^{-n} < \infty$, while $\mathfrak{f}|D|^{-s}$ is zero for $s > n$ and is infinite for $s < n$.

We have enough information now to compute the classical dimension of \mathbb{T}^3 directly. First of all, $D^2 = -(\sigma \cdot \delta)^2 = (-\delta_1^2 - \delta_2^2 - \delta_3^2)$ can be diagonalized in the orthonormal basis of \mathcal{H} given by

$$\psi_r^+ := \begin{pmatrix} \underline{u}^r \\ 0 \end{pmatrix}, \quad \psi_r^- := \begin{pmatrix} 0 \\ \underline{u}^r \end{pmatrix}, \quad r \in \mathbb{Z}^3.$$

Indeed, $D^2 \psi_r^\pm = 4\pi^2 |r|^2 \psi_r^\pm$, and therefore $|D| \psi_r^\pm = 2\pi |r| \psi_r^\pm$. There are two zero modes, ψ_0^+ and ψ_0^- , which we ignore when dealing with $|D|^{-1}$. We compute

$$\mathfrak{f}|D|^{-s} = 2 \lim_{R \rightarrow \infty} \frac{1}{3 \log R} \sum_{1 \leq |r| \leq R} (2\pi |r|)^{-s} = \lim_{R \rightarrow \infty} \frac{2}{3 \log R} \int_1^R \frac{4\pi \rho^2 d\rho}{(2\pi \rho)^s}, \quad (6)$$

which is zero for $s > 3$, diverges for $s < 3$ and equals $1/3\pi^2$ for $s = 3$; so indeed the classical dimension is 3, as expected.

The geometrical apparatus outlined here (Dirac operator, spinor space, cyclic cocycles, volume form, charge conjugation, dimension) of course works the same in the commutative case; to be precise, this is the geometry of compact spin manifolds. In Refs. 13 and 21, the example of the Riemann sphere is done in full detail, and the general theory is laid out in Part III of Ref. 13. In other words, the classical geometry of Riemannian spin manifolds can be rewritten in a purely operatorial language; but in that new language, many other geometries also appear, that may be called spin geometries over noncommutative spaces.

2. Rules and procedures for NC geometries

We now explain more systematically how the various pieces of the geometrical apparatus fit together. We also need to see how it can serve as the basis for other models of physical interest. One of the most striking of these was a phenomenological Yang–Mills–Higgs model put forward by Connes and Lott,^{19,22,23} in order to incorporate symmetry breaking at the geometrical level,²⁴ and later developed by several others.^{25–34} A detailed review of this approach to the Standard Model is given in our Ref. 33.

2.1. The ground rules for spin geometries

We define a noncommutative spin geometry by a list of *terms* and *conditions*.

The terms form a package $\mathcal{G} = (\mathcal{A}, \mathcal{H}, D, C, \chi)$, where $(\mathcal{A}, \mathcal{H}, D)$ is a *spectral triple*.³⁵ This means that \mathcal{A} is an algebra, represented as operators on a Hilbert space \mathcal{H} , and that D is an (unbounded) selfadjoint operator on \mathcal{H} such that $\ker D$ is finite-dimensional and D^{-1} is compact on $\mathcal{H} \ominus \ker D$, and also that $[D, a]$ is bounded for each $a \in \mathcal{A}$. The operator C is an antiunitary conjugation on \mathcal{H} such that $b \mapsto Cb^*C^{-1}$ is a representation of the opposite algebra \mathcal{A}° which commutes with \mathcal{A} , that is, $[a, Cb^*C^{-1}] = 0$ for all $a, b \in \mathcal{A}$. Finally, for χ there are two cases: in the odd case $\chi = 1$, and in the even case χ is a grading operator, that is, a selfadjoint operator such that $\chi^2 = 1$. In the even case, the algebra acts evenly, $\chi a = a\chi$, while the operator D is odd, $\chi D = -D\chi$.

There are seven conditions to satisfy,⁴ which we now list.

1. *Classical dimension*: there is a nonnegative integer n , that is odd or even according as $\chi = 1$ or not, such that $\sum_{k \leq N} s_k(D)^{-n} \sim C \log N$ as $N \rightarrow \infty$, with $0 < C < \infty$. We write $f|D|^{-n} := C$. If \mathcal{A} and \mathcal{H} are finite-dimensional, we set $n = 0$.
2. *Regularity*: the bounded operators a and $[D, a]$, for any $a \in \mathcal{A}$, belong to the domain of smoothness $\bigcap_{k=1}^{\infty} \text{Dom}(\delta^k)$ of the derivation $\delta(T) := [|D|, T]$.
3. *Finiteness*: the space of smooth vectors $\mathcal{H}^\infty := \bigcap_{k=1}^{\infty} \text{Dom}(D^k)$ is a finite projective left module over the pre- C^* -algebra \mathcal{A} . It carries an \mathcal{A} -valued inner product $(\cdot | \cdot)$ implicitly defined by $f(\phi | \psi) |D|^{-n} = \langle \phi | \psi \rangle$.
4. *Reality*: the conjugation C satisfies $C^2 = \pm 1$, $CD = \pm DC$, and $C\chi = \pm \chi C$ in the even case, where the signs are given by the following tables:

$n \bmod 8$	0	2	4	6	$n \bmod 8$	1	3	5	7
$C^2 = \pm 1$	+	-	-	+	$C^2 = \pm 1$	+	-	-	+
$CD = \pm DC$	+	+	+	+	$CD = \pm DC$	-	+	-	+
$C\chi = \pm \chi C$	+	-	+	-					

5. *First order*: $[[D, a], Cb^*C^{-1}] = 0$ for all $a, b \in \mathcal{A}$.
6. *Orientation*: there is a Hochschild n -cycle $\mathbf{c} \in (\mathcal{A} \otimes \mathcal{A}^\circ) \otimes \mathcal{A}^{\otimes n}$, $b\mathbf{c} = 0$, whose representative on \mathcal{H} satisfies $\pi_D(\mathbf{c}) = \chi$.
7. *Poincaré duality*: The Fredholm index of the operator D yields a nondegenerate intersection form on the K -theory of the algebra \mathcal{A} .

We refer to Ref. 4 and Section 10.5 of Ref. 13 for a full discussion of these conditions and their implications. Here we just make a few remarks.

To produce a “compact NC space”, we demand that \mathcal{A} have a unit 1 (think of the constant function 1 on a manifold), and that D^{-1} be compact outside $\ker D$. This may be weakened in the “locally compact” (i.e., nonunital) case by asking instead that $(1 + D^2)^{-1/2}$ be compact. We no longer need to suppose that D is any sort of Dirac operator.

The dimension condition is obtained from a version of Weyl’s formula linking the dimension and volume of a compact manifold to the growth of the eigenvalues of the Laplacian. For spin manifolds, the Lichnerowicz formula $D^2 = \Delta + \frac{1}{4}s$ says that D^2 is a generalized Laplacian on spinors.

The regularity condition, in the commutative case, expresses the smoothness of the coordinate functions a ; this can be seen by working out the symbols of $\delta^k([D, a])$ with pseudodifferential calculus. Another method, developed by Rennie,³⁶ also gives the smoothness of the functions. For the NC tori, the regularity condition imposes the fast decrease of the Fourier series coefficients.

The finiteness condition asks that the space \mathcal{H}^∞ of smooth vectors be either \mathcal{A}^N , a direct sum of several copies of \mathcal{A} (so operators and vectors have the same smoothness properties), or at least of the form $p\mathcal{A}^N$ for some projector $p \in M_N(\mathcal{A})$. For the NC tori example with $\mathcal{A} = \mathbb{T}_\theta^n$, we took $\mathcal{H}^\infty = \mathcal{A}^N$ where $N = 2^m$ when $n = 2m$ or $2m + 1$. In the commutative case, \mathcal{H}^∞ is the space of smooth sections of the spinor bundle.

The tables for the reality condition signal an underlying action of the real Clifford algebra of \mathbb{R}^n , complete with mod-8 periodicity, that indeed gives rise to the charge conjugation operator C : for the full story, see Chapter 9 of Ref. 13. In short, the sign tables arise from the product rules for gamma matrices.

The Poincaré duality condition is a reinterpretation, in K -theory language, of the usual pairing of differential forms of complementary degrees on an oriented manifold: see Section VI.4 of Ref. 19.

Two geometries \mathcal{G}_1 and \mathcal{G}_2 give rise to a product geometry \mathcal{G} , with $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\chi_1 \otimes 1 + 1 \otimes \chi_2$ if, say, \mathcal{G}_1 is even. We take $D = D_1 \otimes 1 + \chi_1 \otimes D_2$. The product conjugation is worked out on a case-by-case basis.³⁷

2.2. Finite geometries

Let us look at a “baby geometry” of classical dimension zero, where \mathcal{A} is a finite-dimensional matrix algebra (stable under taking Hermitian conjugates), acting on a finite-dimensional Hilbert space \mathcal{H} . Then D is a matrix, too, with a finite number of eigenvalues.

This means that any scheme of approximation of a higher-dimensional space by a set of finite matrix algebras is not a straightforward matter. A popular model is the so-called fuzzy sphere,³⁸ where the algebra $C^\infty(\mathbb{S}^2)$ is approximated by algebras $M_{2j+1}(\mathbb{C})$ obtained from the Moyal product on the phase space \mathbb{S}^2 . The issue is how to control the limit $j \rightarrow \infty$ when the classical dimension jumps from 0 to 2; it is fair to say that this problem is still open.

It is instructive, even so, to consider the “two-point space” over the (commutative!) algebra $\mathcal{A} = \mathbb{C}^2$, with $\mathcal{H} = \mathbb{C}^2$ also. We use $\chi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as the grading operator, so that \mathcal{A} must act by diagonal 2×2 matrices. Since $n = 0$, the tables give $C^2 = +1$ and $C\chi = \chi C$, so C must be just complex conjugation. The selfadjoint odd operator D is of the form

$$D = \begin{pmatrix} 0 & m^* \\ m & 0 \end{pmatrix} \quad \text{for some } m \in \mathbb{C}. \quad (7)$$

But $D = CDC = \begin{pmatrix} 0 & m \\ m^* & 0 \end{pmatrix}$, so m is real. We may as well suppose that $m > 0$. The distance between the two points (the pure states of \mathcal{A}) is found from

$$[D, a] = \begin{pmatrix} 0 & -m(a_1 - a_2) \\ m(a_1 - a_2) & 0 \end{pmatrix},$$

so that $[D, a]^2 = -m^2(a_1 - a_2)^2 1_2$, and thus $\|[D, a]\| = m|a_1 - a_2|$. The maximum value of $|a_1 - a_2|$ when $\|[D, a]\| \leq 1$ equals $1/m$. This already indicates that m is an inverse distance, so it can be regarded as a mass.

A more elaborate proposal, arising from the Connes–Lott models,^{22,26,29,33} is to use a baby geometry to stand for the Yukawa terms in the Standard Model Lagrangian (at the classical level). This can be thought of —when combined with the spacetime geometry— as a useful test case of what a realistic NC geometry would look like.

The Hilbert space for the “baby Yukawa” model has a basis labelled by Weyl fermions. The conjugation C exchanges the particle and antiparticle subspaces, and reduces D to the act on each of these separately. We can again split with χ , according to chirality; on the particle subspace, D then looks like (7), where now m is a matrix of lepton and quark masses, incorporating the Cabibbo–Kobayashi–Maskawa mixing.

The algebra of this model is dictated by the gauge symmetry of the Standard Model. We would like a matrix algebra whose unitary group is $U(1) \times SU(2) \times SU(3)$, but there is none available. The best we can manage is to use the local gauge group $U(1) \times SU(2) \times U(3)$, and later eliminate the extra $U(1)$ factor; it turns out that this can be done by anomaly cancellation.³¹ The corresponding algebra is¹⁹

$$\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}),$$

where \mathbb{H} denotes the quaternions. (Had we tried instead to use $\mathbb{H} \oplus M_3(\mathbb{C})$, whose unitary group is $SU(2) \times U(3)$, we would have obtained a model whose leptons are not colour singlets.)

The final step in building the baby Yukawa model is the choice of the representation of \mathcal{A} on the Hilbert space; this involves a delicate balancing, with different prescriptions for the lepton and quark sectors. For the details of the construction, see Ref. 33.

A complete analysis of the finite spectral triples that satisfy the requirements for a noncommutative spin geometry is available.^{39,40} One finds, in essence, that when Poincaré duality is taken into account, the baby Yukawa model is the simplest case that satisfies all the above conditions.

2.3. Gauging the spin geometries

Gauge potentials are “1-forms” that give Hermitian operators:

$$A := \sum_j a_j [D, b_j] \quad \text{with} \quad A^* = A.$$

By 1-forms we mean finite sums of the indicated kind. Since $a [D, b] c = a [D, bc] - ab [D, c]$, such 1-forms make up an \mathcal{A} -bimodule of operators on \mathcal{H} , which we call Ω_D^1 . The gauging rule for the operator D is then

$$D \mapsto D + A + CAC^{-1}.$$

Fermionic action terms are schematically like $I(\Psi) = \langle \Psi | (D + A + CAC^{-1}) \Psi \rangle$.

If we apply to \mathcal{A} an inner automorphism $a \mapsto uau^*$, where u is a unitary element of \mathcal{A} , there is a corresponding unitary operator on \mathcal{H} given by $U := uCuC^{-1}$. Since CuC^{-1} commutes with \mathcal{A} , the operator U implements the same inner automorphism. Moreover, U commutes with C (by design) and with χ . The upshot is that the package $\mathcal{G} = (\mathcal{A}, \mathcal{H}, D, C, \chi)$ is unitarily equivalent to $(\mathcal{A}, \mathcal{H}, UDU^{-1}, C, \chi)$, and so may be regarded as the same spin geometry. We now compute

$$\begin{aligned} UDU^{-1} &= CuC^{-1}uDu^*Cu^*C^{-1} = CuC^{-1}(D + u[D, u^*])Cu^*C^{-1} \\ &= CuC^{-1}DCu^*C^{-1} + u[D, u^*] = D + CuC^{-1}[D, Cu^*C^{-1}] + u[D, u^*] \\ &= D + u[D, u^*] + Cu[D, u^*]C^{-1}. \end{aligned}$$

What this means is that the gauge potential $A := u[D, u^*]$ is trivial, in that it does not alter the geometry.

For a more general selfadjoint $A \in \Omega_D^1$, the recipe

$$\nabla c =: [D, c] + Ac$$

defines a covariant derivative on \mathcal{A} with values in Ω_D^1 . We then get a new selfadjoint operator \tilde{D} on \mathcal{H} from the Leibniz rule

$$\begin{aligned} \tilde{D}(b\psi) &:= (\nabla b)\psi + bD\psi + bC(\nabla 1)C^{-1}\psi \\ &= [D, b]\psi + Ab\psi + bD\psi + bCAC^{-1}\psi \\ &= (D + A + CAC^{-1})b\psi. \end{aligned}$$

The new geometry is generally not unitarily equivalent to the original one, so we have arrived at a wider notion of gauge equivalence of geometries.

In fact, one can go much further, generalizing the operator $\nabla: \mathcal{A} \rightarrow \Omega_D^1$ by introducing connections on finitely generated projective (right) \mathcal{A} -modules, that play the role of vector bundles in NCG. Given such a module \mathcal{E} , a connection is a linear map $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1$ that satisfies a Leibniz rule

$$\nabla(sa) = (\nabla s)a + s \otimes [D, a].$$

Working with this connection involves changing the Hilbert space to $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \bar{\mathcal{E}}$ and the algebra to $\mathcal{B} := \text{End}_{\mathcal{A}} \mathcal{E}$, namely, the operators on \mathcal{E} that commute with the right action of \mathcal{A} . Such a relation among algebras is called Morita equivalence. We refer to Ref. 21 for more information on that, and to Ref. 41 for a full discussion of connections. The point is that Morita equivalence of algebras is directly linked to gauge equivalence of geometries.

3. QFT on a noncommutative space

We now consider, albeit briefly, what we can learn about quantum field theory by using the technology of noncommutative spaces and geometries. The emergence of the NC tori in compactified Matrix theories shows that, in principle, one should be prepared to “go noncommutative” at the outset. Indeed, there has been a recent revival of interest in the idea of noncommuting spacetime coordinates. Doplicher, Fredenhagen and Roberts⁴² provided strong evidence for a Moyal product structure on the spacetime variables (noncommutative \mathbb{R}^4). Recently, Seiberg and Witten, pulling various strands together, have discussed the pervasive presence of noncommutative tori in string theory (see Ref. 3 and references therein).

The first to follow up the suggestion that noncommuting coordinates might help to mollify UV divergences was Filk,⁴³ who analyzed how the Moyal product affected path-integral divergences; his analysis, based on general topological properties of the Feynman graphs, found no overall improvement in the UV behaviour. Our own approach⁵ starts with canonical quantization of fermions, treating the gauge field as an external classical source; we also found no better overall UV behaviour. By quantizing the nonlinear theory by the dual way of treating the fermions with Fock space techniques, and the gauge bosons by functional integration, this conclusion can be shown to be generally valid. Other recent investigations^{6,44} reinforce this view. The question is why this “no-go” theorem should hold.

3.1. Quantization of NC spaces

We illustrate the project by setting up the free Dirac equation on a noncommutative 3-torus. Writing $p_j := -i \delta_j$, we recall from (5) that

$$D = -i \sigma \cdot \delta = \sigma \cdot p,$$

so that the solutions of the equations

$$(i\partial_t - \sigma \cdot p)\psi_R = 0, \quad (i\partial_t + \sigma \cdot p)\psi_L = 0$$

represent Weyl neutrinos on the spinor space $\mathcal{H} = \mathcal{H}_\tau \oplus \mathcal{H}_\tau$. If one wishes, one can introduce a mass term which couples these neutrinos:

$$(i\partial_t - \sigma \cdot p)\psi_R = m\psi_L, \quad (i\partial_t + \sigma \cdot p)\psi_L = m\psi_R,$$

or, more compactly,

$$\begin{pmatrix} 0 & i\partial_t + \sigma \cdot p \\ i\partial_t - \sigma \cdot p & 0 \end{pmatrix} \psi = m\psi, \quad \text{with} \quad \psi := \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}. \quad (8)$$

Write $E(p) := (p_1^2 + p_2^2 + p_3^2 + m^2)^{1/2}$. This is the positive square root of a positive operator on \mathcal{H}_τ , and we can introduce two more positive operators,

$$(\sigma p) \equiv \sigma^\mu p_\mu := E + \sigma \cdot p, \quad (\bar{\sigma} p) := E - \sigma \cdot p.$$

Their positive square roots, in turn, are given by

$$\sqrt{(\sigma p)} \text{ or } \sqrt{(\bar{\sigma} p)} := \frac{1}{\sqrt{2}} \{ (E + m)^{1/2} \pm (E - m)^{1/2} \sigma \cdot p \}.$$

The equation (8) then has “plane-wave” solutions of the form

$$\psi_{E,r} = e^{-iEt} u^r \begin{pmatrix} \sqrt{(\bar{\sigma} r)} \xi \\ \sqrt{(\sigma r)} \xi \end{pmatrix},$$

with $r \in \mathbb{Z}^3$ and $\xi \in \mathbb{C}^2$, $|\xi| = 1$.

In brief, the usual Dirac equation calculations go through in $\mathbb{R}_t \times \mathbb{T}_\theta^3$, on replacing positive functions with positive operators.

The *phase* of the (free) Dirac operator is $F := D|D|^{-1} = D(D^2)^{-1/2}$. On each two-dimensional subspace of \mathcal{H} spanned by $\{\psi_r^+, \psi_r^-\}$, for $r = (r_1, r_2, r_3) \in \mathbb{Z}^3$, we can express D and F in terms of Pauli matrices:

$$D = 2\pi \sigma \cdot r, \quad |D| = 2\pi |r|, \quad F = \frac{\sigma \cdot r}{|r|}. \quad (9)$$

The eigenvalues are then the same as for the ordinary Dirac operator on the ordinary torus (with untwisted boundary conditions).

In order to proceed to Fock space, we split the “one-particle space” as $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, in positive- and negative-frequency solutions of the free Dirac equation. That is to say, the grading operator for this splitting is the phase operator F . The space of solutions \mathcal{H} should be regarded as a *real* Hilbert space, on which we are free to impose a complex scalar product. The interpretation of the negative-frequency solutions as antiparticles may be implemented by taking the scalar product to be

$$\langle \psi | \phi \rangle_F := \langle \psi_+ | \phi_+ \rangle + \langle \phi_- | \psi_- \rangle.$$

Any one-particle operator A on \mathcal{H} can be written in block form:

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}.$$

Its odd and even parts can be distinguished with the phase operator:

$$\begin{aligned} A_+ &:= \begin{pmatrix} A_{++} & 0 \\ 0 & A_{--} \end{pmatrix} = \frac{1}{2}(A + FAF), \\ A_- &:= \begin{pmatrix} 0 & A_{+-} \\ A_{-+} & 0 \end{pmatrix} = \frac{1}{2}(A - FAF) = \frac{1}{2}F[F, A]. \end{aligned}$$

Choose any orthonormal bases $\{\phi_k\}$ for \mathcal{H}^+ and $\{\psi_k\}$ for \mathcal{H}^- , and let $b_k := b(\phi_k)$, $d_k := d(\psi_k)$ denote corresponding annihilation operators, together with the creation operators b_k^\dagger , d_k^\dagger . The quantum counterpart of A on Fock space is

$$\mathbf{A} := b^\dagger A_{++} b + b^\dagger A_{+-} d^\dagger + d A_{-+} b + :d A_{--} d^\dagger:, \quad (10)$$

where $b^\dagger A_{++} b := \sum_{j,k} b_k^\dagger \langle \phi_k | A_{++} \phi_j \rangle b_j$, $b^\dagger A_{+-} d^\dagger := \sum_{j,k} b_k^\dagger \langle \phi_k | A_{+-} \psi_j \rangle d_j^\dagger$, and so on; the last term is normal-ordered. This “second-quantization” rule corresponds to the infinitesimal spin representation on Fock space, developed originally by Shale and Stinespring.⁴⁵ It is independent of the orthonormal bases used, but makes sense only when A_{+-} and A_{-+} are Hilbert–Schmidt or, equivalently, only when $[F, A]$ is Hilbert–Schmidt. The details of this quantization recipe can be found, for instance, in Refs. 13,46.

In $1+1$ dimensions, this is enough, since we can usually guarantee that $[F, A]$ lies in the Hilbert–Schmidt class \mathcal{L}^2 for pertinent operators A ; and then A is implementable on Fock space by (10). In other words, normal ordering is sufficient to regularize the theory.

For higher dimensions, we seek to implement gauge transformations on Fock space. Instead of seeking out the most general gauge potentials, we only examine the trivial vector bundles of rank one (that is, $\mathcal{E} = \mathcal{A}$), with gauge potentials in Ω_D^1 . If $A = \sum_j a_j [D, b_j]$, then (since $FD = DF$), we find that

$$[F, A] = \sum_j [F, a_j] [D, b_j] + a_j [D, [F, b_j]],$$

and so the issue is to determine when $[F, a] \in \mathcal{L}^2$ for all $a \in \mathcal{A}$.

The triple $(\mathcal{A}, \mathcal{H}, F)$ is called a *Fredholm module* over the algebra \mathcal{A} . In general, such an object consists of an algebra \mathcal{A} represented on a Hilbert space \mathcal{H} , together with a selfadjoint operator F such that $F^2 = 1$ and $[F, a]$ is compact for each $a \in \mathcal{A}$. When F is the phase of the operator D defining a spectral triple, these properties are clear, except possibly the compactness of $[F, a]$. However, a straightforward spectral-theory calculation⁴⁷ shows that, when $a^* = -a$ so that $[F, a]$ is selfadjoint, the operator inequality

$$-||[D, a]|| |D|^{-1} \leq [F, a] \leq ||[D, a]|| |D|^{-1} \quad (11)$$

holds, so that $[F, a]$ is compact since $|D|^{-1}$ is compact and $[D, a]$ is bounded.

In $1+1$ quantum field theory, the Schwinger term can be written as

$$\alpha(A, B) = \frac{1}{8} \operatorname{Tr}(F[F, A][F, B]),$$

if A and B represent infinitesimal gauge transformations, for which $[F, A]$ and $[F, B]$ lie in \mathcal{L}^2 . As well as being a 2-cocycle for Lie algebra cohomology, it is also a cyclic 1-cocycle.⁴⁸ Indeed, its coboundary is

$$b\alpha(a, b, c) = \frac{1}{8} \text{Tr}(F[F, ab][F, c] - F[F, a][F, bc] + F[F, ca][F, b]) = 0.$$

To go beyond the $1 + 1$ case, we start by noting that if $a \in C^\infty(M)$ for M a spin manifold of dimension n , and if F is the phase of the Dirac operator, then $[F, a] \in \mathcal{L}^p$ for $p > n$; this can be proved with pseudodifferential calculus. Here \mathcal{L}^p means the Schatten p -class; a compact operator A lies in \mathcal{L}^p if $\sum_k s_k(A)^p$ converges or, equivalently, if $|A|^p$ is traceclass. More generally, if $[F, a_j] \in \mathcal{L}^p$ for $j = 1, \dots, k$, then⁴⁹

$$[F, a_1][F, a_2] \dots [F, a_k] \in \mathcal{L}^{p/k}$$

so that the following *Chern character* makes sense, provided that each $[F, a_j]$ lies in \mathcal{L}^{n+1} :

$$\tau_F(a_0, a_1, \dots, a_n) := \frac{1}{2} \text{Tr}(\chi F [F, a_0] [F, a_1] \dots [F, a_n]). \quad (12)$$

This is an $(n + 1)$ -linear form on \mathcal{A} , which is cyclic—that is, moving $[F, a_n]$ to the position before $[F, a_0]$ changes it only by the sign $(-1)^n$ of the corresponding cyclic permutation—and one can check that its coboundary vanishes: τ_F is a *cyclic n -cocycle*. We say that the Fredholm module $(\mathcal{A}, \mathcal{H}, F)$ has “quantum dimension” n if every $[F, a]$ lies in \mathcal{L}^{n+1} .

3.2. The quantum dimension of a noncommutative torus

For the NC torus \mathbb{T}_θ^n , the question of UV divergences comes down to this: is it possible to improve the implementability of gauge transformations by showing that $[F, a] \in \mathcal{L}^p$ for some $p \leq n$ and all $a \in \mathcal{A}$? To answer this question, we compute the quantum dimension of a 3-torus.

We examine the effect of $[F, a]$ on the two-dimensional subspace spanned by $\{\psi_r^+, \psi_r^-\}$ for some fixed $r \in \mathbb{Z}^3$. From (9), we know that $F\psi_r^\pm = |r|^{-1}(\sigma \cdot r)\psi_r^\pm$. On the other hand, if $a = \sum_s a_s u^s$, the left multiplication operator a does not act block diagonally; instead, using (3), we find that $a\psi_r^\pm = \sum_s \lambda(s, r) a_s \psi_{r+s}^\pm$. Thus,

$$[F, a]\psi_r^\pm = \sum_s \lambda(s, r) a_s \left(\frac{\sigma \cdot (r + s)}{|r + s|} - \frac{\sigma \cdot r}{|r|} \right) \psi_{r+s}^\pm.$$

In the special case $a = u^s$, the operator $[F, u^s]^*[F, u^s]$ is diagonal, so

$$\begin{aligned} [F, u^s]^*[F, u^s]\psi_r^\pm &= \lambda(r + s, s) \lambda(s, r) \left(\frac{\sigma \cdot (r + s)}{|r + s|} - \frac{\sigma \cdot r}{|r|} \right)^2 \psi_r^\pm \\ &= 2 \left(1 - \frac{(r + s) \cdot r}{|r + s| |r|} \right) \psi_r^\pm. \end{aligned}$$

Therefore, if $p \geq 1$,

$$\begin{aligned}
\|[F, u^s]\|_p^p &= \|[F, u^s]^*[F, u^s]\|_p^{p/2} = 2 \cdot 2^{p/2} \sum_{r \neq 0} \left(1 - \frac{(r+s) \cdot r}{|r+s||r|}\right)^{p/2} \\
&= 2 \sum_{r \neq 0} \left\{ 2 - 2 \left(1 + \frac{r \cdot s}{|r|^2}\right) \left(1 + 2 \frac{r \cdot s}{|r|^2} + \frac{|s|^2}{|r|^2}\right)^{-1/2} \right\}^{p/2} \\
&= 2 \sum_{r \neq 0} \left\{ \frac{|r \times s|^2}{|r|^4} + O(|r|^{-3}) \right\}^{p/2}.
\end{aligned}$$

We conclude that $[F, u^s] \in \mathcal{L}^p$ iff $\sum_{r \neq 0} |r|^{-p}$ converges, iff $\int_1^\infty \rho^{2-p} d\rho$ converges, if and only if $p > 3$.

On the other hand, a similar computation⁵ shows that $\|[F, a]\|_4^4 < \infty$ for all $a \in \mathbb{T}_\theta^3$; thus $[F, a] \in \mathcal{L}^4$ in all cases. We conclude that the quantum dimension of \mathbb{T}_θ^3 is greater than 2 but not greater than 3, so it equals 3. Recalling now the calculation (6), that shows that the classical dimension is also 3, we see that both dimensions coincide.

It is noteworthy that the parameter θ enters the above calculation only through the cocycle $\lambda(s, r)$, which gets estimated by its absolute value and ceases to matter. Therefore, the overall UV behaviour is the same for noncommutative as for commutative tori, and no improvement due to noncommutativity can be obtained.

3.3. The NC index theorem prevents UV improvement

At first, one might think that the previous “no-go theorem” could be a particular feature of canonical quantization, or an accidental property of the tori. But we may recall that Filk⁴³ found the same result by a different argument for noncommutative \mathbb{R}^4 , and that Krajewski and Wulkenhaar^{6,40} also found the same UV divergences for a Yang–Mills field over the 4-torus, by path-integral methods. There have been a few claims of UV improvement in some models, but these either lie outside our framework of spin geometry, or treat zero-dimensional approximations only.

To see why it must be so, we call upon one of the deepest results in noncommutative geometry, which establishes the relation between the noncommutative integral defined by a generalized Dirac operator D and the Chern character of its phase operator F . This is the Hauptsatz of Connes, stating that both of these integrals take the same values on “volume forms”: see p. 308 of Ref. 19. The precise statement is that, for a geometry of classical dimension n , the following equality holds:

$$\frac{1}{2} \operatorname{Tr}(\chi F \sum_j [F, a_0^j] [F, a_1^j] \dots [F, a_n^j]) = \oint \chi \sum_j a_0^j [D, a_1^j] \dots [D, a_n^j] |D|^{-n}, \quad (13)$$

whenever $\sum_j a_0^j \otimes a_1^j \otimes \dots \otimes a_n^j$ is a Hochschild n -cycle over \mathcal{A} .

The proof of this theorem is a long story; this is not surprising, because it includes the Atiyah–Singer index theorem for the case $\mathcal{A} = C^\infty(M)$, as is made plain in Ref. 50. It is also a particular case of the local index theorem in NCG developed by Connes and Moscovici.⁵¹ By interpolating a one-parameter family of cyclic cocycles between both sides of (13), one can construct a McKean–Singer type of proof: see Chapter 10 of our Ref. 13.

The right hand side reduces to an ordinary integral in the commutative case. In fact, for the Dirac operator (and untwisted spin structure) on \mathbb{T}^3 , it can be shown that

$$\oint a_0 [D, a_1][D, a_2][D, a_2] |D|^{-3} = \frac{i}{3\pi^2} \int_{\mathbb{T}^3} a_0 da_1 \wedge da_2 \wedge da_3.$$

This is achieved by Connes’ trace theorem,⁵² which establishes that the noncommutative integral of a pseudodifferential operator is proportional to its Wodzicki residue, which in turn is given by the ordinary integral of a local density. Direct computation of the Chern character is more difficult, since F is given by a singular integral operator, and the trace is, in principle, a highly nonlocal integral of a suitable kernel. For \mathbb{T}^3 (or \mathbb{R}^3), Langmann⁵³ obtained

$$\frac{1}{2} \text{Tr}(\chi^F [F, a_0][F, a_1][F, a_2][F, a_3]) = \frac{i}{3\pi^2} \int_{\mathbb{T}^3} a_0 da_1 \wedge da_2 \wedge da_3,$$

which nicely corroborates the index theorem, and indeed suffices for toral geometries.

Now, if \mathbf{c} is the Hochschild n -cycle providing the orientation of a given spin geometry, for which $\pi_D(\mathbf{c}) = \chi$, we can plug it into the formula (13). Using an obvious notation, we get

$$\tau_F(\mathbf{c}) = \oint \chi \pi_D(\mathbf{c}) |D|^{-n} = \oint |D|^{-n} = C > 0,$$

since $\chi^2 = 1$. If the quantum dimension were less than n , the Chern character τ_F would have to be of the form $(-2/n) S \tau'_F$, where τ'_F is the analogous cyclic $(n-2)$ -cocycle, and S is the periodicity operator that promotes cyclic $(n-2)$ -cocycles to cyclic n -cocycles; this follows from the periodicity theorem in cyclic cohomology.¹⁹ However, another consequence of the periodicity theorem is that promoted cyclic cocycles are trivial in Hochschild cohomology, which means that $\tau_F(\mathbf{c})$ would vanish. So the Hauptsatz guarantees that the quantum dimension is at least n .

On the other hand, the estimate (11) shows that $[F, a] \in \mathcal{L}^p$ whenever $|D|^{-1} \in \mathcal{L}^p$, which happens for all $p > n$. Thus, the quantum dimension is exactly n , so it coincides with the classical dimension in all cases. In other words, the no-go theorem is an inescapable feature of noncommutative geometry.

4. Hopf algebras in noncommutative geometry

The local index theorem of Connes and Moscovici⁵¹ expresses the character τ_F of a Fredholm module obtained from a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ as a complicated sum of noncommutative integrals (which are “local”) involving products of operators of the form $[D^2, \dots [D^2[D, a]] \dots]$ and a compensating $|D|^{-m}$. These can be computed, in principle, as residues of certain zeta functions, but the computations turned out to be very extensive, even for geometries of dimension 1. In seeking to streamline the calculations, Connes and Moscovici identified a Hopf algebra that governs the terms appearing in the index formula.⁸ This Hopf algebra, which we shall briefly describe, is commutative but highly noncocommutative: it is not a familiar one from the literature on quantum groups.

A short while previously, Kreimer⁷ had found a similar Hopf algebra in a seemingly different context, namely, the combinatorial structure of the Zimmermann forest formula for renormalizing integrals corresponding to Feynman graphs! It turns out that both Hopf algebras are closely related.⁹ Subsequent work has shown that these Hopf algebra structures point to a new and deep relationship between NCG and QFT.¹⁰ We cannot do justice to it here, so we shall merely sketch the origins of these Kreimer–Connes–Moscovici algebras and show how they are related.

4.1. An example of diffeomorphism-invariant geometry

To deal with gravity in the NCG framework, one seeks to understand the geometrical features that are invariant under diffeomorphisms.⁴ For instance, Landi and Rovelli⁵⁵ have explored how the eigenvalues of the Dirac operator provide the natural variables for such a theory.

As a first step, we can consider how to study the invariants of an oriented manifold M under the action of a subgroup Γ of the diffeomorphism group $\text{Diff}(M)$. The orbit space M/Γ is the leaf space of a foliation, but could have an unpleasant topology (think of the Kronecker foliation of a 2-torus under rotations at an irrational angle). In NCG, where we use the algebra instead of the space, this problem is remedied by taking the “crossed product” $\mathcal{A}_0 := C^\infty(M) \rtimes \Gamma$ as the natural algebra of coordinates. The crossed product is defined as the algebra generated by the functions in $C^\infty(M)$ and a set of unitaries $\{V_\psi : \psi \in \Gamma\}$, subject to the relations

$$V_\psi h V_\psi^\dagger = h^\psi, \quad \text{where} \quad h^\psi(x) := h(\psi^{-1}(x)).$$

Such an algebra is highly noncommutative, and may lack easily constructed representations; there will often be no Γ -invariant measure that would help to build a Hilbert space (Γ could be, say, the full diffeomorphism group). The way out of this dilemma is to replace M by the (oriented) frame bundle $F \rightarrow M$, and to find a Γ -invariant measure on F . To see how this works, consider the one-dimensional case. As usual, we take M to be compact, so it is just the circle $M = \mathbb{S}^1$ with local coordinate θ . The frame bundle is a cylinder, with vertical

coordinate $y = e^{-s}$, and any $\phi \in \Gamma \subseteq \text{Diff}^+(\mathbb{S}^1)$ acts on F by

$$\tilde{\phi}(\theta, s) = (\phi(\theta), s - \log \phi'(\theta)).$$

Then it is easy to see that each $\tilde{\phi}$ preserves the measure $e^s ds d\theta$ on F . (This means that F can be depicted as the funnel obtained by revolution of the graph of the logarithm.) We can now use the Hilbert space $\mathcal{H} := L^2(F, e^s ds d\theta)$.

The corresponding algebra $\mathcal{A} := C_0^\infty(F) \rtimes \tilde{\Gamma}$ acts on \mathcal{H} and is generated by elements $\{fU_\psi^\dagger : f \in C_0^\infty(F), \psi \in \Gamma\}$, with the product rule

$$(fU_\psi^\dagger)(gU_\phi^\dagger) := f(g \circ \tilde{\psi})U_{\phi\psi}^\dagger.$$

The horizontal and vertical vector fields

$$X := e^{-s} \frac{\partial}{\partial \theta}, \quad Y := -\frac{\partial}{\partial s}$$

serve to define an operator D by solving the equation $D|D| = Y^2 + X$; this is the D that enters into the local index computation. To compute anything explicitly, we need to know how X and Y interact with the unitary generators U_ψ^\dagger . It turns out that $[Y, U_\psi^\dagger] = 0$, but the horizontal vector fields obey a more complicated formula:

$$X(fU_\psi^\dagger gU_\phi^\dagger) = (Xf)U_\psi^\dagger gU_\phi^\dagger + fU_\psi^\dagger X(gU_\phi^\dagger) + e^{-s} \frac{\psi''(\theta)}{\psi'(\theta)} fU_\psi^\dagger Y(gU_\phi^\dagger).$$

Write $\lambda_1(fU_\psi^\dagger) := (e^{-s}\psi''(\theta)/\psi'(\theta))fU_\psi^\dagger$. This is a derivation of the algebra \mathcal{A} . The previous formula can now be abbreviated as

$$X(ab) = X(a)b + aX(b) + \lambda_1(a)Y(b), \quad a, b \in \mathcal{A}. \quad (14)$$

It is obvious that $[Y, X] = X$ as vector fields on F , and this relation transfers to their action on \mathcal{A} ; moreover, $[Y, \lambda_1] = \lambda_1$. But X is not quite a derivation and X, Y and λ_1 do not close to a Lie algebra acting on \mathcal{A} . In fact, if $\lambda_2 := [X, \lambda_1]$, we find that

$$\lambda_2(fU_\psi^\dagger) = e^{-2s} \frac{\psi'\psi''' - \psi''^2}{\psi'^2} fU_\psi^\dagger.$$

In general, if we introduce

$$\lambda_n(fU_\psi^\dagger) := e^{-ns} \frac{\partial^n}{\partial \theta^n} (\log \psi'(\theta)) fU_\psi^\dagger,$$

then $X, Y, \lambda_1, \dots, \lambda_n, \dots$ closes to a Lie algebra, with the commutation relations

$$[Y, X] = X, \quad [X, \lambda_n] = \lambda_{n+1}, \quad [Y, \lambda_n] = n\lambda_n, \quad [\lambda_m, \lambda_n] = 0.$$

We can rewrite (14) and the Leibniz rule for Y in the language of coproducts:

$$\begin{aligned} \Delta Y &= Y \otimes 1 + 1 \otimes Y, \\ \Delta X &= X \otimes 1 + 1 \otimes X + \lambda_1 \otimes Y. \end{aligned}$$

The analogous rules for the $\lambda_1, \lambda_2, \lambda_3$ may be found from these and the commutation rules, bearing in mind that Δ is a homomorphism of algebras:

$$\begin{aligned}\Delta\lambda_1 &= \lambda_1 \otimes 1 + 1 \otimes \lambda_1, \\ \Delta\lambda_2 &= \lambda_2 \otimes 1 + 1 \otimes \lambda_2 + \lambda_1 \otimes \lambda_1, \\ \Delta\lambda_3 &= \lambda_3 \otimes 1 + 1 \otimes \lambda_3 + 3\lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_1 + \lambda_1^2 \otimes \lambda_1.\end{aligned}\quad (15)$$

We can continue recursively, using $\Delta\lambda_{n+1} = [\Delta X, \Delta\lambda_n]$. From (15) it is clear that the elements λ_n (without X or Y) generate a Hopf algebra that is commutative but by no means cocommutative.

Any Hopf algebra comes equipped with an antipode S , which is the unique linear map from the Hopf algebra into itself such that both $m(S \otimes \text{id})\Delta$ and $m(\text{id} \otimes S)\Delta$ are equal to the map taking 1 to 1 and all other generators to 0. (Here m denotes the algebra product.) More explicitly, if $\Delta\alpha := \sum_j \alpha'_j \otimes \alpha''_j$ is the coproduct of a nontrivial generator α , then $\sum_j S(\alpha'_j)\alpha''_j = \sum_j \alpha'_j S(\alpha''_j) = 0$. With the coproduct and product in hand, the antipode can be determined; see Refs. 8,13 or Ref. 56 for the details. For example,

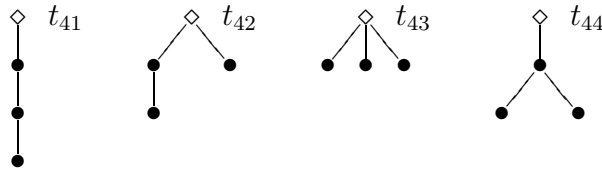
$$S(\lambda_1) = -\lambda_1, \quad S(\lambda_2) = -\lambda_2 + \lambda_1^2, \quad S(\lambda_3) = -\lambda_3 + 4\lambda_1\lambda_2 - 2\lambda_1^3.$$

With these tools, one can continue to compute the index formula in low-dimensional cases.⁸

4.2. Nested subdivergences and the forest formula

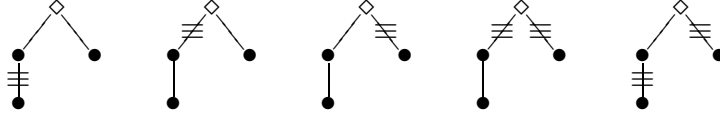
And now for something completely different. Suppose that we wish to deal with a multiloop Feynman graph with superficially divergent subgraphs, which we hope to renormalize by subtracting appropriate counterterms. If there are no overlapping divergences, the subdivergences form a family of subgraphs that are either nested or disjoint; such a family is called a forest. The counterterms may be assigned by Zimmermann's forest formula,⁵⁴ which is a sum over all forests in the given diagram and constitutes a recursive rule for the several levels of nesting. Kreimer⁷ discovered that Zimmermann's procedure is encoded in a Hopf algebra, whose elements are called "rooted trees".⁹

Given a diagram with only nested or disjoint subdivergences, the *root* of the tree, depicted at the top, represents the full diagram. The leaves of the tree are subdivergences that include no proper subdivergences, and ascending links indicate the intermediate nestings. Nodes that are linked only through a higher node represent disjoint subdivergences. Here are all the rooted trees with four nodes:



These rooted trees generate a commutative *algebra*, whose unit 1 corresponds to the empty tree. The product is denoted by juxtaposition, and the sum is a formal one (it corresponds to a sum of integrals for several Feynman graphs). We make it a *Hopf algebra* by introducing a coproduct and identifying an antipode.

To get the coproduct, we need to see how the tree may be cut by lopping off one or more branches from the root part (which we call the *trunk*), without ever cutting a piece already separated from the root. For the rooted tree denoted t_{42} , here are the allowable cuts:



For each allowable cut c of a rooted tree T , we denote the trunk by $R_c(T)$, and the product of the pruned branches by $P_c(T)$. The coproduct satisfies $\Delta(1) := 1 \otimes 1$, and is given on the nontrivial generators by

$$\Delta T := T \otimes 1 + 1 \otimes T + \sum_c P_c(T) \otimes R_c(T).$$

The first two terms may be absorbed in the sum by adding an “empty cut” whose trunk is the whole tree, and a “full cut” that prunes the whole tree. Here, for example, is $\Delta(t_{42})$:

$$\Delta\left(\begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}\right) = \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} \otimes \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} \otimes \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} \otimes \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} \otimes \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array} \otimes \begin{array}{c} \diamond \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}$$

To determine the antipode, note first that $1 = m(S \otimes \text{id})(1 \otimes 1) = S(1)$, and that $m(S \otimes \text{id})\Delta(T) = S(T) + T + \sum_c S(P_c(T)) R_c(T)$, and thus

$$S(T) = -T - \sum_c S(P_c(T)) R_c(T).$$

This recursive recipe for S turns out to give precisely the forest formula! That is proved in Ref. 56. Thus the lore of counterterms may be reduced to understanding the recipe for the above coproduct of trees.

4.3. How both Hopf algebras are related

To find the relationship between Kreimer’s Hopf algebra and that of Connes and Moscovici, we must do a little gardening. There is a unique rooted tree with one node (the solitary root, called t_1) and only one with two nodes (call it t_2); there are two rooted trees with three nodes, called t_{31} (three nodes in a chain) and t_{32} (two leaves sprouting from the root), and we have already seen the four rooted trees with four nodes. There is an easy way to produce new trees from old,⁹ called “natural growth”: for any tree T , let $N(T)$ be the *sum* of all the trees formed by

adding one new branch and leaf at *each* node of T . Thus $N(t_{31}) = t_{41} + t_{42} + t_{43}$, $N(t_{32}) = 2t_{42} + t_{44}$, and so on.

Introduce $\delta_1 := t_1$, $\delta_2 := N(t_1) = t_2$, $\delta_3 := N(t_2) = t_{31} + t_{32}$, and in general $\delta_{n+1} := N(\delta_n)$. For example, $\delta_4 = t_{41} + 3t_{42} + t_{43} + t_{44}$. If we compute the coproducts of these sums of trees, we find that

$$\begin{aligned}\Delta\delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1, \\ \Delta\delta_2 &= \delta_2 \otimes 1 + 1 \otimes \delta_2 + \delta_1 \otimes \delta_1, \\ \Delta\delta_3 &= \delta_3 \otimes 1 + 1 \otimes \delta_3 + 3\delta_1 \otimes \delta_2 + \delta_2 \otimes \delta_1 + \delta_1^2 \otimes \delta_1,\end{aligned}$$

and so on. These are exactly the same formulas as (15)!

Why so? The connection lies in observing that the natural growth operator satisfies a Leibniz rule:

$$N(T_1 T_2) = N(T_1) T_2 + T_1 N(T_2),$$

since, given two juxtaposed trees T_1 and T_2 , we may hang the extra node on either one. We may express this by forming the (abelian) Lie algebra generated by the trees and introducing an extra generator X by declaring that $[X, T] := N(T)$. Moreover, the number of nodes $\#T$ in a tree T gives a \mathbb{Z} -grading on the algebra of trees, since $\#(T_1 T_2) = \#T_1 + \#T_2$. We can then add another generator Y to the Lie algebra by declaring that $[Y, T] := (\#T) T$. Finally, observe that

$$\begin{aligned}[[Y, X], T] &= [[Y, T], X] + [Y, [X, T]] = (\#T) [T, X] + [Y, N(T)] \\ &= -(\#T) N(T) + (\#T + 1) N(T) = N(T) = [X, T],\end{aligned}$$

so that the Lie algebra closes with $[Y, X] = X$.

We can compute the coproduct $\Delta(N(T))$, too. All we have to do is to grow an extra leaf on T and then cut the resulting trees in every allowable way. If the new branch is not cut in this process, then it belongs to either a pruned branch or to the trunk that remains after a cut has been made on the original tree T ; this amounts to $(N \otimes \text{id})\Delta(T) + (\text{id} \otimes N)\Delta(T)$. On the other hand, if the new branch is cut, the new leaf contributes a solitary node δ_1 to P_c ; the new leaf must have been attached to the trunk $R_c(T)$ at any one of the latter's nodes. Since $(\#R_c)R_c = [Y, R_c]$, the terms wherein the new leaf is cut amount to $[\delta_1 \otimes Y, \Delta(T)]$. In total,

$$\Delta(N(T)) = (N \otimes \text{id})\Delta(T) + (\text{id} \otimes N)\Delta(T) + [\delta_1 \otimes Y, \Delta(T)].$$

Thus, since $\Delta[X, T] = [\Delta(X), \Delta(T)]$ must hold, we get

$$\Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y.$$

Also, it is easy to see that $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$.

The conclusion is that, with these extra generators X and Y , the Hopf subalgebra generated by X , Y and the various δ_n is isomorphic to the Connes–Moscovici algebra.

The moral of the story is that Hopf algebras provide a new and useful entry point for noncommutative geometry into the business of renormalization. The hope that this will shed new light on QFT can already be justified.^{10,57}

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